ASYMPTOTICS OF SEMIGROUPS GENERATED BY OPERATOR MATRICES

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Dedicated to Rainer Nagel on the occasion of his 65^{th} birthday.

Abstract. We survey some known results about generator property of operator matrices with diagonal or coupled domain. Further, we use basic properties of the convolution of operator-valued mappings in order to obtain stability results for such semigroups.

1. Operator matrices with diagonal domain

While tackling abstract problems that are related to concrete initial–boundary value problems with dynamical boundary conditions and/or with coupled systems of PDE's, it is common that one has to check whether an operator matrix

$$\mathbf{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

generates a C_0-semigroup on a suitable product Banach space, see e.g. [18], [17], and [16]. To fix the ideas, let us impose the following.

Assumptions 1.1.

(1) X and Y are Banach spaces.
(2) $A : D(A) \subset X \to X$ is linear and closed.
(3) $D : D(D) \subset Y \to Y$ is linear and closed.
(4) $B : D(B) \subset Y \to X$ is linear, with $D(D) \subset D(B)$.
(5) $C : D(C) \subset X \to Y$ is linear, with $D(A) \subset D(C)$.

Let us first deal with operator matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with diagonal domain $D(A) \times D(D)$.

Then, it is an elementary exercise to check that A and D generate a semigroup on X and Y, respectively, if and only if the operator matrix generates a semigroup on the product space $X \times Y$. Accordingly, taking into account standard perturbation results for generators of strongly continuous or analytic semigroups, the following can be proven using the techniques of [17, § 3]. Throughout the paper we define by

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$[D(H)]$ the Banach space obtained by endowing the domain of a closed operator H on a Banach space by its graph norm.

Theorem 1.2. *Under the Assumptions 1.1 the following assertions hold for the operator matrix* $\mathbf{A}$ *defined in* (1.1) *with diagonal domain* $D(\mathbf{A}) := D(A) \times D(D)$ *on the product space* $\mathbf{X} := X \times Y$.

(1) *Let*
- $B \in \mathcal{L}([D(D)], [D(A)])$, *or else* $B \in \mathcal{L}(Y, X)$, *and moreover*
- $C \in \mathcal{L}([D(A)], [D(D)])$, *or else* $C \in \mathcal{L}(X, Y)$.

Then A *and* D *both generate* C_0*-semigroups* $(e^{tA})_{t\geq 0}$ *on* X *and* $(e^{tD})_{t\geq 0}$ *on* Y, *respectively, if and only if* $\mathbf{A}$ *generates a* C_0*-semigroup* $(e^{t\mathbf{A}})_{t\geq 0}$ *on* $\mathbf{X}$.

(2) *Let*
- $C \in \mathcal{L}([D(A)], Y)$ *and* $B \in \mathcal{L}(Y, X)$, *or else*
- $B \in \mathcal{L}([D(D)], X)$ *and* $C \in \mathcal{L}(X, Y)$.

Then A *and* D *both generate analytic semigroups* $(e^{tA})_{t\geq 0}$ *on* X *and* $(e^{tD})_{t\geq 0}$ *on* Y, *respectively, if and only if* $\mathbf{A}$ *generates an analytic semigroup* $(e^{t\mathbf{A}})_{t\geq 0}$ *on* $\mathbf{X}$.

(3) *Let* A *and* D *both generate analytic semigroups* $(e^{tA})_{t\geq 0}$ *on* X *and* $(e^{tD})_{t\geq 0}$ *on* Y, *respectively. Let both these semigroups have analyticity angle* $\delta \in (0, \frac{\pi}{2}]$. *If there exists* $\alpha \in (0,1)$ *such that*
- $B \in \mathcal{L}([D(A)], [D(D), Y]_\alpha)$ *and*
- $C \in \mathcal{L}([D(D)], [D(A), X]_\alpha)$,

then $\mathbf{A}$ *generates an analytic semigroup* $(e^{t\mathbf{A}})_{t\geq 0}$ *of angle* $\delta \in (0, \frac{\pi}{2}]$ *on* $\mathbf{X}$. *Conversely, if* $\mathbf{A}$ *generates an analytic semigroup of angle* $\delta \in (0, \frac{\pi}{2}]$ *on* $\mathbf{X}$ *and*
$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(D(\mathbf{A}), [D(\mathbf{A}), \mathbf{X}]_\alpha)$$
for some $\alpha \in (0,1)$, *then also* A *and* D *generate semigroups of angle* δ *on* X *and* Y, *respectively.*

If any of the above assertions hold with $B = 0$, *then*
$$R(t) := \int_0^t e^{(t-s)D} C e^{sA} ds$$
is well-defined as a bounded operator from X *to* Y *for all* $t \geq 0$ *and there holds*
$$e^{t\mathbf{A}} = \begin{pmatrix} e^{tA} & 0 \\ R(t) & e^{tD} \end{pmatrix}, \qquad t \geq 0. \tag{1.2}$$
Likewise, if instead $C = 0$, *then the semigroup generated by* $\mathbf{A}$ *has the form*
$$e^{t\mathbf{A}} = \begin{pmatrix} e^{tA} & S(t) \\ 0 & e^{tD} \end{pmatrix}, \qquad t \geq 0,$$
where
$$S(t) := \int_0^t e^{(t-s)A} B e^{sD} ds \in \mathcal{L}(Y, X), \qquad t \geq 0.$$

Proof. The assertions (1) and (2) as well as (1.2) have been obtained in [17, Prop. 3.1, Cor. 3.2, and Cor. 3.3]. In order to prove (3), simply decompose $\mathbf{A}$ as
$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{1.3}$$

and observe that the first addend on the right hand side has diagonal domain $D(A) \times D(C)$ and generates an analytic semigroup, so that the complex interpolation space is given by $[D(\mathbf{A}), \mathbf{X}]_\alpha = [D(A), X]_\alpha \times [D(D), Y]_\alpha$ for all $\alpha \in (0,1)$. Now the second addend on the right hand side of (1.3) is a bounded linear operator from $[D(\mathbf{A})]$ to $[D(\mathbf{A}), \mathbf{X}]_\alpha$. The claim follows by a perturbation result due to Desch–Schappacher, cf. [6]. $\square$

Remark 1.3. 1) By the bounded perturbation theorem one obtains that if M, ϵ are constants such that $\|e^{tA}\| \le Me^{\epsilon t}$ and $\|e^{tD}\| \le Me^{\epsilon t}$, $t \ge 0$, then

$$\|e^{t\mathbf{A}}\| \le Me^{(\epsilon + M \max\{\|B\|, \|C\|\})t}, \qquad t \ge 0,$$

whenever B, C are bounded operators. In particular, $(e^{t\mathbf{A}})_{t \ge 0}$ is uniformly exponentially stable provided that $(e^{tA})_{t \ge 0}$ and $(e^{tD})_{t \ge 0}$ are uniformly exponentially stable, too, and that moreover $M \max\{\|B\|, \|C\|\} < -\epsilon$. We are going to sharpen this result in Proposition 1.8.

2) Let $B = 0$ and C be bounded. If $A = 0$ and D is invertible, then $R(t) = \int_0^t e^{sD} C ds = D^{-1}(e^{tD} - I)C$. Thus, $(e^{t\mathbf{A}})_{t \ge 0}$ is bounded if and only if $(e^{tD})_{t \ge 0}$ is bounded. Likewise, if $D = 0$ and A is invertible, then $(e^{t\mathbf{A}})_{t \ge 0}$ is bounded if and only if $(e^{tA})_{t \ge 0}$ is bounded. In either case if $(e^{t\mathbf{A}})_{t \ge 0}$ is uniformly exponentially stable, then $C = 0$. Analogous assertions hold if B is bounded and $C = 0$.

In the remainder of this section we are going to show that the matrix structure of our problem allows to prove better results. Recall that by the Datko–Pazy theorem a C_0-semigroup on a Banach space E is uniformly exponentially stable if and only if it is of class $L^1(\mathbb{R}_+, \mathcal{L}_s(E))$.

If the operator matrix $\mathbf{A}$ is upper or lower triangular, the form of $(R(t))_{t \ge 0}$ and $(S(t))_{t \ge 0}$ allows us to apply known results on convolutions of operator valued mappings. In the following we state most results in the case of $B = 0$ and $C \in \mathcal{L}(X, Y)$, but of course analogous results hold whenever $C = 0$ and $B \in \mathcal{L}(Y, X)$.

Proposition 1.4. *Let Theorem 1.2 apply with $B = 0$ and $C \in \mathcal{L}(X, Y)$. Assume $(e^{tD})_{t \ge 0}$ to be uniformly exponentially stable. Then the following hold.*

(1) If for some $x \in X$ the orbit $(e^{tA}x)_{t \ge 0}$ is bounded, then the orbit $(R(t)x)_{t \ge 0}$ is bounded as well.
(2) Under the assumptions of (1), if additionally the orbit $(e^{tA}x)_{t \ge 0}$ is asymptotically almost periodic, then the orbit $(R(t)x)_{t \ge 0}$ is asymptotically almost periodic as well.
(3) If $\lim_{t \to \infty} e^{tA}x$ exists, then $\lim_{t \to \infty} R(t)x = D^{-1}C \lim_{t \to \infty} e^{tA}x$.
(4) If $(e^{tA})_{t \ge 0}$ is uniformly exponentially stable, then $(e^{t\mathbf{A}})_{t \ge 0}$ is uniformly exponentially stable as well.

Proof. Observe that for all $x \in X$ $R(t)x$ can be seen as the convolution $T * f$, where $(T(t))_{t \ge 0} := (e^{tD})_{t \ge 0}$ is a strongly continuous family of bounded linear operators on Y and for all $x \in X$ the mapping $f := \left(Ce^{\cdot A}x\right)$ is of class $L^1_{loc}(\mathbb{R}_+, Y)$. Now it follows from the Young inequality for operator-valued functions, cf. [1, Prop. 1.3.5], that $T * f \in L^r(\mathbb{R}_+, Y)$ whenever $T \in L^p(\mathbb{R}_+, \mathcal{L}(Y))$ and $f \in L^q(\mathbb{R}_+, Y)$ for $1 \le p, q, r \le \infty$ such that $p^{-1} + q^{-1} = 1 + r^{-1}$.

Thus, the Young inequality for $p = 1$ and $q = \infty$ or $q = 1$ yields (1) and (4), respectively. The assertions (2) and (3) follow by [1, Prop. 5.6.1.c)–d)]. $\square$

Proposition 1.5. *Let Theorem 1.2 apply with $B = 0$ and $C \in \mathcal{L}(X, Y)$. Assume $(e^{tA})_{t\geq 0}$ and $(e^{tD})_{t\geq 0}$ to be uniformly exponentially stable and bounded, respectively. Then the following hold.*

1. *The semigroup $(e^{t\mathbf{A}})_{t\geq 0}$ is bounded as well.*
2. *If additionally $(e^{tD})_{t\geq 0}$ is asymptotically almost periodic, then $(R(t))_{t\geq 0}$ is asymptotically almost periodic.*
3. *If $\lim_{t\to\infty} e^{tD}$ exists (resp., exists and is equal 0) in the strong operator topology, then $\lim_{t\to\infty} e^{t\mathbf{A}}$ exists (resp., exists and is equal 0) in the strong operator topology as well.*

Proof. The assertions follow from [1, Prop. 5.6.4], again by considering $R(t)x = T * f$, $x \in X$. □

Remark 1.6. Under the assumptions of Proposition 1.4.(1) or Proposition 1.5.(1) let the semigroups $(e^{tA})_{t\geq 0}$ and $(e^{tD})_{t\geq 0}$ have uniform bounds M_1 and M_2, respectively. Let moreover $\epsilon < 0$ such that

$$\|e^{tA}\| \leq M_1 e^{\epsilon t} \qquad \text{or} \qquad \|e^{tD}\| \leq M_2 e^{\epsilon t}, \qquad t \geq 0.$$

By the Young inequality we obtain that $\|R(\cdot)x\|_\infty \leq -\frac{M_1 M_2 \|C\|}{\epsilon}\|x\|$, $x \in X$, i.e.,

$$\left\| e^{t\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq \max\left\{ M_1, -\frac{M_1 M_2 \|C\|}{\epsilon} \right\} \|x\| + M_2 \|y\|, \qquad t \geq 0,\ x \in X,\ y \in Y.$$

If moreover both

$$\|e^{tA}\| \leq M_1 e^{\epsilon t} \qquad \text{and} \qquad \|e^{tD}\| \leq M_2 e^{\epsilon t}, \qquad t \geq 0,$$

hold, then for all $t \geq 0$ and $x \in X$

$$\begin{aligned} \|R(t)x\| &\leq M_1 M_2 \|C\| \|x\| e^{\epsilon t} \int_0^t ds = M_1 M_2 \|C\| t e^{\epsilon t} \|x\| \\ &\leq -\frac{M_1 M_2 \|C\|}{\epsilon e} \|x\|. \end{aligned}$$

In particular

$$\left\| e^{t\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq \max\left\{ M_1, -\frac{M_1 M_2 \|C\|}{\epsilon e} \right\} \|x\| + M_2 \|y\|, \qquad t \geq 0,\ x \in X,\ y \in Y.$$

Asymptotical results can also be obtained by imposing so-called non-resonance conditions, cf. [1, § 5.6].

Proposition 1.7. *Let Theorem 1.2 apply with $B = 0$ and $C \in \mathcal{L}(X, Y)$. Assume that $i\mathbb{R} \cap \sigma(A) \cap \sigma(D) = \emptyset$. If moreover for the vector $x \in X$ the orbit $(e^{tA}x)_{t\geq 0}$ is bounded and $(e^{tD})_{t\geq 0}$ is bounded, too, then the following hold.*

1. *If $(e^{tD})_{t\geq 0}$ is analytic, then the orbit $(R(t)x)_{t\geq 0}$ is bounded.*
2. *Let the orbit $(R(t)x)_{t\geq 0}$ be bounded. If $(e^{tD})_{t\geq 0}$ is asymptotically almost periodic and moreover the orbit $(e^{tA}x)_{t\geq 0}$ is asymptotically almost periodic, then $(R(t)x)_{t\geq 0}$ is asymptotically almost periodic as well.*
3. *Let the orbit $(R(t)x)_{t\geq 0}$ be bounded. If $\lim_{t\to\infty} e^{tD}$ exists (resp., exists and is equal 0) in the strong operator topology, and if $\lim_{t\to\infty} e^{tA}x$ exists (resp., exists and is equal 0), then $\lim_{t\to\infty} R(t)x$ exists (resp., exists and is equal 0) as well.*

Proof. As in the proofs of previous results, we write $R(t)x = T * f$, where $(T(t))_{t\geq 0} := (e^{tD})_{t\geq 0}$ and for all $f := \left(Ce^{\cdot A}x\right)$. Observe that the Laplace transform $\hat{f}(\lambda)$ of f is given by $CR(\lambda, A)$, $\mathrm{Re}\lambda > 0$: thus the half-line spectrum $\mathrm{sp}(f)$ of f, defined as in [1, § 4.4], is given by $\{\eta \in \mathbb{R} : i\eta \in \sigma(A)\}$. Then the claims follow from [1, Thm. 5.6.5 and Thm. 5.6.6]. □

Finally, we are able to prove an asymptotical result for the semigroup generated by the complete (i.e., with $B \neq 0 \neq C$) operator matrix. The following result should be compared with Remark 1.3.

Proposition 1.8. *Let $M_1, M_2 \geq 1$ and $\epsilon_1, \epsilon_2 < 0$ be constants such that*

$$\|e^{tA}\| \leq M_1 e^{\epsilon_1 t} \qquad \text{and} \qquad \|e^{tD}\| \leq M_2 e^{\epsilon_2 t}, \qquad t \geq 0. \tag{1.4}$$

Let B and C be bounded operators and assume that $M_1 M_2 \|B\|\|C\| < \epsilon_1 \epsilon_2$. Then the semigroup generated by the complete matrix $\mathbf{A}$ is uniformly exponentially stable.

Proof. The semigroup $(e^{t\mathbf{A}})_{t\geq 0}$ generated by the complete matrix is given by the Dyson–Phillips series

$$\sum_{k=0}^{\infty} S_k(t), \qquad t \geq 0,$$

where $(S_0(t))_{t\geq 0}$ is the semigroup

$$\begin{pmatrix} e^{tA} & 0 \\ R(t) & e^{tD} \end{pmatrix}, \quad t \geq 0, \qquad \text{generated by} \qquad \begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

and

$$S_k(t) := \int_0^t S_0(t-s) \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} S_{k-1}(s) ds, \qquad t \geq 0,\ k = 1, 2, \ldots.$$

Let $k = 1, 2, \ldots$. If we denote by $S_k^{(ij)}(t)$ the (i,j)-entry of the operator matrix $S_k(t)$, $t \geq 0$, $1 \leq i, j \leq 2$, then we have

$$\begin{aligned} S_k(t) &= \int_0^t \begin{pmatrix} e^{(t-s)A} & 0 \\ R(t-s) & e^{(t-s)D} \end{pmatrix} \begin{pmatrix} BS_{k-1}^{(21)}(s) & BS_{k-1}^{(22)}(s) \\ 0 & 0 \end{pmatrix} ds \\ &= \int_0^t \begin{pmatrix} e^{(t-s)A} BS_{k-1}^{(21)}(s) & e^{(t-s)A} BS_{k-1}^{(22)}(s) \\ R(t-s) BS_{k-1}^{(21)}(s) & R(t-s) BS_{k-1}^{(22)}(s) \end{pmatrix} ds \\ &= \begin{pmatrix} e^{\cdot A} * BS_{k-1}^{(21)}(\cdot) & e^{\cdot A} * BS_{k-1}^{(22)}(\cdot) \\ R(\cdot) * BS_{k-1}^{(21)}(\cdot) & R(\cdot) * BS_{k-1}^{(22)}(\cdot) \end{pmatrix}. \end{aligned}$$

We are going to prove that the estimates

$$\begin{aligned} \|S_k^{(11)}(\cdot)x\|_{L^1(\mathbb{R}_+, X)} &\leq \frac{M_1^{k+1} M_2^k \|C\|^k \|B\|^k}{|\epsilon_1|^{k+1} |\epsilon_2|^k} \|x\|, \qquad x \in X, \\ \|S_k^{(12)}(\cdot)y\|_{L^1(\mathbb{R}_+, X)} &\leq \frac{M_1^k M_2^k \|C\|^{k-1} \|B\|^k}{|\epsilon_1|^k |\epsilon_2|^k} \|y\|, \qquad y \in Y, \\ \|S_k^{(21)}(\cdot)x\|_{L^1(\mathbb{R}_+, Y)} &\leq \frac{M_1^{k+1} M_2^{k+1} \|C\|^{k+1} \|B\|^k}{|\epsilon_1|^{k+1} |\epsilon_2|^{k+1}} \|x\|, \qquad x \in X, \\ \|S_k^{(22)}(\cdot)y\|_{L^1(\mathbb{R}_+, Y)} &\leq \frac{M_1^k M_2^{k+1} \|C\|^k \|B\|^k}{|\epsilon_1|^k |\epsilon_2|^{k+1}} \|y\|, \qquad y \in Y, \end{aligned}$$

hold for all $k \in \mathbb{N}$. By (1.4) one obtains that $\|e^{\cdot A}x\|_{L^1} \leq -\frac{M_1}{\epsilon_1}\|x\|$ and $\|e^{\cdot D}y\|_{L^1} \leq -\frac{M_2}{\epsilon_2}\|x\|$, and by the Young inequality also $\|R(\cdot)x\|_{L^1} = \|e^{\cdot D}*Ce^{\cdot A}x\|_{L^1} \leq \frac{M_1M_2\|C\|}{\epsilon_1\epsilon_2}\|x\|$, and this proves that the above inequalities hold for $k=0$. Let them now hold for k. Then for $k+1$ one applies the Young inequality and obtains

$$\begin{aligned}\|S^{(11)}_{k+1}(\cdot)x\|_{L^1} &= \|e^{\cdot A} * BS^{(21)}_k(\cdot)x\|_{L^1} \leq \|e^{\cdot A}\|_{L^1}\|BS^{(21)}_k(\cdot)x\|_{L^1} \\ &\leq \frac{M_1^{k+2}M_2^{k+1}\|C\|^{k+1}\|B\|^{k+1}}{|\epsilon_1|^{k+2}|\epsilon_2|^{k+1}}\|x\|.\end{aligned}$$

The other three estimates can be proven likewise.

Let us now prove the proposition's claim. We can assume that $C \neq 0$, otherwise the claim follows directly by Proposition 1.4.(4). Let $\|B\|\|C\| < \frac{\epsilon_1\epsilon_2}{M_1M_2}$. Then the series $\sum_{k=0}^{\infty}\left(\frac{M_1M_2\|B\|\|C\|}{\epsilon_1\epsilon_2}\right)^k$ converges, and by the dominated convergence theorem one has for all $x \in X$ and $y \in Y$,

$$\begin{aligned}\int_0^\infty \|T(t)\begin{pmatrix}x\\y\end{pmatrix}\|dt &\leq \int_0^\infty \left(\sum_{k=0}^\infty \|S_k(t)\begin{pmatrix}x\\y\end{pmatrix}\|\right)dt \\ &\leq \sum_{k=0}^\infty \|S^{(11)}_k(\cdot)x\|_{L^1} + \sum_{k=0}^\infty \|S^{(12)}_k(\cdot)y\|_{L^1} \\ &\quad + \sum_{k=0}^\infty \|S^{(21)}_k(\cdot)x\|_{L^1} + \sum_{k=0}^\infty \|S^{(22)}_k(\cdot)y\|_{L^1} \\ &= \sum_{k=0}^\infty \left(\frac{M_1M_2\|B\|\|C\|}{\epsilon_1\epsilon_2}\right)^k \cdot \\ &\quad \cdot \left(\frac{M_1}{\epsilon_1}\|x\| + \frac{1}{\|C\|}\|y\| + \frac{M_1M_2\|C\|}{\epsilon_1\epsilon_2}\|x\| + \frac{M_2}{\epsilon_2}\|y\|\right) \\ &< \infty.\end{aligned}$$

By the theorem of Datko–Pazy this concludes the proof. □

Remark 1.9. Clearly, the above criterion is particularly useful when $M_1 \neq M_2$ and $\epsilon_1 \neq \epsilon_2$. However, already in the simple case of $A = D$ and $B = \alpha C$, $\alpha \in \mathbb{R}$, one obtains by means of Proposition 1.8 a sharper result than in Remark 1.3.(1).

2. Operator matrices with non-diagonal domain

Motivated by applications to initial-boundary value problems (see, e.g., [3, 14, 4, 10, 5, 15]), we want to deduce results similar to those of Section 1 for the same operator matrix $\mathbf{A}$, defined however on a different, *coupled* domain

$$D(\mathbf{A}) := \left\{\begin{pmatrix}u\\x\end{pmatrix} \in D(A) \times (D(D) \cap \partial Y) \;:\; Lu = x\right\}, \tag{2.1}$$

where L is a *boundary operator* operator from X to ∂Y. Here ∂Y is a suitable Banach space continuously imbedded in the *boundary space* $\partial X := Y$. More precisely, in the remainder of this section we replace the Assumptions 1.1 by the following.

Assumptions 2.1.

(1) $X, \partial X, \partial Y$ are Banach spaces such that $\partial Y \hookrightarrow \partial X$.
(2) $A : D(A) \subset X \to X$ is linear.

(3) *L is surjective from $D(A)$ to ∂Y.*
(4) *$A_0 := A_{|\ker(L)}$ has nonempty resolvent set.*
(5) *$\begin{pmatrix} A \\ L \end{pmatrix}$ is closed (as an operator from X to $X \times \partial Y$).*
(6) *$D : D(D) \to \partial X$ is linear and $D(D) \cap \partial Y$ is a linear subspace of ∂X.*
(7) *$B : D(B) \subset Y \to X$ is linear, with $D(D) \subset D(B)$.*
(8) *$C : D(C) \subset X \to Y$ is linear, with $D(A) \subset D(C)$.*

Remark 2.2. We denote by $[D(A)_L]$ the Banach space obtained by endowing $D(A)$ with the graph norm of the closed operator $\binom{A}{L}$.

In several situations one already knows that $\binom{A}{L}$ is closed as an operator from X to $X \times \partial X$. It is clear that if $\binom{A}{L}$ is closed as an operator from X to $X \times \partial X$, then it is also closed as an operator from X to $X \times \partial Y$.

Let us consider the abstract eigenvalue Dirichlet problem

$$\text{(ADP)} \qquad \begin{cases} Au &= \lambda u, \\ Lu &= x. \end{cases}$$

The following is a slight modification of a result due to Greiner, cf. [4, Lemma 2.3] and [16, Lemma 3.2].

Lemma 2.3. *Under the Assumptions 2.1, the problem* (ADP) *admits a unique solution $u := D_\lambda^{A,L} x$ for all $x \in \partial Y$ and $\lambda \in \rho(A_0)$. Moreover, the* Dirichlet operator *$D_\lambda^{A,L}$ is bounded from ∂Y to Z for every Banach space Z satisfying $D(A^\infty) \subset Z \hookrightarrow X$. In particular, $D_\lambda^{A,L} \in \mathcal{L}(\partial Y, [D(A)_L])$.*

Remark 2.4. It follows by definition that also A_0 is closed and further $[D(A_0)] \hookrightarrow [D(A)_L]$ as well as $[D(\mathbf{A})] \hookrightarrow [D(A)_L] \times [D(D)]$.

Lemma 2.5. *Let $\lambda \in \rho(A_0)$. Then the factorization*

$$\mathbf{A} - \lambda = \mathbf{A}_\lambda \mathbf{M}_\lambda := \begin{pmatrix} A_0 - \lambda & B \\ C & D + CD_\lambda^{A,L} - \lambda \end{pmatrix} \begin{pmatrix} I_X & -D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix}$$

holds. Here $D + CD_\lambda^{A,L}$ has domain $D(D) \cap \partial Y$.

An analogue of the above factorization is the starting point of the discussion in [7, 8], and can be proven likewise, cf. also [12]. Unlike in the setting of [12], $\mathbf{M}_\lambda$ is in general an *unbounded* operator on $\mathbf{X}$. We are thus led to impose the following.

Assumptions 2.6.
(1) *C is bounded from $[D(A)_L]$ to ∂X.*
(2) *$D_\lambda^{A,L}$ can be extended for some $\lambda \in \rho(A_0)$ to an operator $\overline{D_\lambda^{A,L}} \in \mathcal{L}(\partial X, W)$, for some Banach space W such that $[D(A)_L] \hookrightarrow W \hookrightarrow X$.*

Lemma 2.7. *Under the Assumptions 2.1 and 2.6 the operator matrix $\mathbf{A} - \lambda$ is similar to*

$$\tilde{\mathbf{A}}_\lambda := \begin{pmatrix} A_0 - \overline{D_\lambda^{A,L}} C - \lambda & B - \overline{D_\lambda^{A,L}} (D + CD_\lambda^{A,L} - \lambda) \\ C & D + CD_\lambda^{A,L} - \lambda \end{pmatrix}, \tag{2.2}$$

with diagonal domain

$$D(\tilde{\mathbf{A}}_\lambda) := D(A_0) \times (D(D) \cap \partial Y),$$

for all $\lambda \in \rho(A_0)$. The similarity transformation is performed by the operator

$$\overline{\mathbf{M}_\lambda} := \begin{pmatrix} I_X & -\overline{D_\lambda^{A,L}} \\ 0 & I_{\partial X} \end{pmatrix},$$

which by Assumptions 2.6 is an isomorphism on $\mathbf{X} = X \times \partial X$.

Corollary 2.8. *Under the Assumptions 2.1 and 2.6, let $\mathbf{A}$ have nonempty resolvent set. Then $\mathbf{A}$ has compact resolvent if and only if the embeddings $[D(A_0)] \hookrightarrow X$ and $[D(D) \cap \partial Y] \hookrightarrow \partial X$ are both compact.*

By Lemma 2.7 the operator matrix with coupled domain $\mathbf{A}$ is a generator on $\mathbf{X}$ if and only if the similar operator matrix $\mathbf{A}_\lambda$ with diagonal domain is a generator on the same space. Several criteria implying this have been already shown, cf. [11, Prop. 4.3] and [4, Cor. 2.8]. The following unifies and generalizes all of them.

Theorem 2.9. *Under the Assumptions 2.1 and 2.6, the following assertions hold.*

(1) *Let $B \in \mathcal{L}(\partial X, X)$ and $D \in \mathcal{L}(\partial X)$. If we can choose $\partial Y = \partial X$, then the operator matrix $\mathbf{A}$ generates a C_0-semigroup on $\mathbf{X}$ if and only if the operator $A_0 - D_\lambda^{A,L} C$ generates a C_0-semigroup on X for some $\lambda \in \rho(A_0)$.*

(2) *Let $C \in \mathcal{L}(X, \partial X)$ and moreover $B \in \mathcal{L}([D(D) \cap \partial X], X)$. Then $\mathbf{A}$ generates an analytic semigroup on $\mathbf{X}$ if and only if for some $\lambda \in \rho(A_0)$ both A_0 and $D + CD_\lambda^{A,L}$ generate analytic semigroups on X and ∂X, respectively.*

(3) *Let A_0 and $D + CD_\lambda^{A,L}$ generate analytic semigroups of angle $\delta \in (0, \frac{\pi}{2}]$ on X and ∂X, respectively. Let for some $0 < \alpha < 1$ the complex interpolation spaces associated to A_0 and $D + CD_\lambda^{A,L}$ satisfy*
   - $\overline{D_\lambda^{A,L}}(\partial X) \hookrightarrow [D(A_0), X]_\alpha$,
   - $B \in \mathcal{L}\left([D(D) \cap \partial Y], [D(A_0), X]_\alpha\right)$, *and further*
   - $C \in \mathcal{L}\left([D(A_0)], [[D(D) \cap \partial Y], \partial X]_\alpha\right)$.

   Then $\mathbf{A}$ generates an analytic semigroup of angle δ on $\mathbf{X}$.

Proof. Take $\lambda \in \rho(A_0)$. By Lemma 2.5.(2) the operator matrix $\mathbf{A} - \lambda$ is similar to $\tilde{\mathbf{A}}_\lambda$ defined in (2.2). Thus, $\mathbf{A}$ is a generator if and only if $\tilde{\mathbf{A}}_\lambda$ is a generator.

(1) By assumption $\overline{D_\lambda^{A,L}} = D_\lambda^{A,L}$. Thus, we can decompose

$$\tilde{\mathbf{A}}_\lambda = \begin{pmatrix} A_0 - D_\lambda^{A,L} C & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & D_\lambda^{A,L}(\lambda - D - CD_\lambda^{A,L}) \\ 0 & D + CD_\lambda^{A,L} - \lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathbf{A}}_\lambda) = D(A_0) \times \partial X$.

Observe that the second operator on the right-hand side is bounded on $\mathbf{X}$ by Lemma 2.3. Since $C \in \mathcal{L}([D(A)_L], \partial X)$, by Remark 2.4 also $C \in \mathcal{L}([D(A_0)], \partial X)$. Now, the claim follows by Theorem 1.2.(1).

(2) We decompose

$$\tilde{\mathbf{A}}_\lambda = \begin{pmatrix} A_0 & B - \overline{D_\lambda^{A,L}}(D + CD_\lambda^{A,L}) \\ 0 & D + CD_\lambda^{A,L} \end{pmatrix} + \begin{pmatrix} -\overline{D_\lambda^{A,L}} C - \lambda & \lambda \overline{D_\lambda^{A,L}} \\ C & -\lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathbf{A}}_\lambda) = D(A_0) \times D(D_\lambda)$.

Since $C \in \mathcal{L}(X, \partial X)$, by Lemma 2.3 the second operator on the right hand side is bounded on $\mathbf{X}$. Hence, by the bounded perturbation theorem $\tilde{\mathbf{A}}_\lambda$ generates an

analytic semigroup on $\mathbf{X}$ if and only if

$$\begin{pmatrix} A_0 & B - \overline{D_\lambda^{A,L}}(D + CD_\lambda^{A,L}) \\ 0 & D + CD_\lambda^{A,L} \end{pmatrix} \qquad \text{with domain} \qquad D(A_0) \times (D(D) \cap \partial Y)$$

generates an analytic semigroup on $\mathbf{X}$. Since $\overline{D_\lambda^{A,L}}(D + CD_\lambda^{A,L}) \in \mathcal{L}([D(D) \cap \partial X], X)$, the claim follows by Lemma 1.2.(2).

(3) We decompose

$$\tilde{\mathbf{A}}_\lambda = \begin{pmatrix} A_0 & 0 \\ 0 & D + CD_\lambda^{A,L} \end{pmatrix} + \begin{pmatrix} -\overline{D_\lambda^{A,L}}C & B - \overline{D_\lambda^{A,L}}(D + CD_\lambda^{A,L}) \\ C & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & \lambda \overline{D_\lambda^{A,L}} \\ 0 & -\lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathbf{A}}_\lambda) = D(A_0) \times (D(D) \cap \partial X)$.

The first addend on the right-hand side generates an analytic semigroup on $\mathbf{X}$ and for $\alpha \in (0,1)$ the corresponding complex interpolation space is $[D(\tilde{\mathbf{A}}_\lambda), \mathbf{X}]_\alpha = [D(A_0), X]_\alpha \times [[D(D) \cap \partial Y], \partial X]_\alpha$. Thus, by assumption the second addend on the right-hand side is bounded from $[D(\tilde{\mathbf{A}}_\lambda)]$ to $[D(\tilde{\mathbf{A}}_\lambda), \mathbf{X}]_\alpha$, while the third one is bounded on $\mathbf{X}$. Hence, by the Desch–Schappacher perturbation theorem (see [6]) the operator matrix $\tilde{\mathbf{A}}_\lambda$ generates an analytic semigroup on $\mathbf{X}$. $\square$

Let $0 \in \rho(A_0)$ and let $C = 0$. Then by Lemma 2.7 the operator $\mathcal{A}$ is similar to

$$\tilde{\mathbf{A}}_0 := \begin{pmatrix} A_0 & B - \overline{D_0^{A,L}}D \\ 0 & D \end{pmatrix}, \tag{2.3}$$

with diagonal domain $D(\tilde{\mathbf{A}}_\lambda) := D(A_0) \times D(D)$. By Theorem 1.2 we conclude that if $\mathbf{A}$ is a generator, then the semigroup has the form

$$e^{t\mathbf{A}} = \begin{pmatrix} e^{tA_0} & S(t) \\ 0 & e^{tD} \end{pmatrix}, \qquad t \geq 0,$$

where $(S(t))_{t \geq 0}$ is a suitable strongly continuous family of convolution operators.

In order to apply the results obtained in Section 1 for triangular operator matrices, for the remainder of this section we impose the following.

Assumption 2.10. *A_0 is invertible and $B - \overline{D_0^{A,L}}D$ is bounded from ∂X to X.*

Remark 2.11. By Remark 1.3.(2), if $C = D = 0$, then under the Assumptions 2.1 and 2.10 we see that $(e^{t\mathbf{A}})_{t \geq 0}$ is bounded if and only $(e^{tA_0})_{t \geq 0}$ is bounded.

We are now in the position to apply the stability results obtained in Section 1, complementing some results in [4, § 5] (where positivity assumptions are essential) and generalizing [5, Thm. 2.7].

Proposition 2.12. *Under the Assumptions 2.1 and 2.10, let A_0 generate a uniformly exponentially stable C_0-semigroup and $C = 0$. If also D generates a C_0-semigroup, then the following hold.*

1. *(1) If for some $y \in Y$ the orbit $(e^{tD}y)_{t \geq 0}$ is bounded, then the orbit $(S(t)y)_{t \geq 0}$ is bounded as well.*
2. *(2) Under the assumptions of (1), if additionally the orbit $(e^{tD}y)_{t \geq 0}$ is asymptotically almost periodic, then the orbit $(S(t)y)_{t \geq 0}$ is asymptotically almost periodic as well.*
3. *(3) If $\lim_{t \to \infty} e^{tD}y$ exists, then $\lim_{t \to \infty} S(t)y = A_0^{-1}B \lim_{t \to \infty} e^{tD}y$.*

(4) If $(e^{tD})_{t\geq 0}$ is uniformly exponentially stable, then $(e^{t\mathbf{A}})_{t\geq 0}$ is uniformly exponentially stable as well.

Such assertions can be directly proved by observing that the assumptions of Proposition 1.4 are satisfied, since in particular the upper-right entry of (2.3) is bounded from ∂X to X. Similarly, from Proposition 1.5 we obtain the following.

Proposition 2.13. *Under the Assumptions 2.1 and 2.10, let A_0 generate a bounded C_0-semigroup and $C = 0$. If D generates a uniformly exponentially stable semigroup, then the following hold.*

(1) The semigroup $(e^{t\mathbf{A}})_{t\geq 0}$ is bounded as well.
(2) If additionally $(e^{tA_0})_{t\geq 0}$ is asymptotically almost periodic, then $(R(t))_{t\geq 0}$ is asymptotically almost periodic.
(3) If $\lim_{t\to\infty} e^{tA_0}$ exists (resp., exists and is equal 0) in the strong operator topology, then $\lim_{t\to\infty} e^{t\mathbf{A}}$ exists (resp., exists and is equal 0) in the strong operator topology as well.
(4) If $(e^{tA_0})_{t\geq 0}$ is uniformly exponentially stable, then $(e^{t\mathbf{A}})_{t\geq 0}$ is uniformly exponentially stable as well.

The following parallels Proposition 1.7. It is comparable to [5, Thm. 2.8].

Proposition 2.14. *Under the Assumptions 2.1 and 2.10, let A_0 generate a bounded C_0-semigroup and $C = 0$. Let further $i\mathbb{R}\cap\sigma(A_0)\cap\sigma(D) = \emptyset$. If the orbit $(e^{tD}y)_{t\geq 0}$ is bounded, then the following hold.*

(1) If $(e^{tA_0})_{t\geq 0}$ is analytic, then the orbit $(S(t)y)_{t\geq 0}$ is bounded.
(2) Let the orbit $(S(t)y)_{t\geq 0}$ be bounded. If $(e^{tA_0})_{t\geq 0}$ is asymptotically almost periodic and moreover the orbit $(e^{tD}y)_{t\geq 0}$ is asymptotically almost periodic, then $(S(t)y)_{t\geq 0}$ is asymptotically almost periodic as well.
(3) Let the orbit $(S(t)y)_{t\geq 0}$ be bounded. If $\lim_{t\to\infty} e^{tA_0}$ exists (resp., exists and is equal 0) in the strong operator topology, and if $\lim_{t\to\infty} e^{tD}y$ exists (resp., exists and is equal 0), then $\lim_{t\to\infty} S(t)y$ exists (resp., exists and is equal 0) as well.

Remark 2.15. Observe that the operator $\overline{D_\lambda^{A,L}}(D-\lambda)$ is always bounded from $[D(D)]$ to X for all $\lambda\in\rho(A_0)$. Thus, if D is an unbounded operator on ∂X, we can still apply the above results to the part of D in $[D(D)]$, and hence to the part of $\mathbf{A}$ in $X\times[D(D)]$.

In several concrete applications it is important to allow abstract boundary feedback operators $C\neq 0$. The following is analogue to Proposition 1.8.

Proposition 2.16. *Under the Assumption 2.1 and 2.10, let $M_1, M_2 \geq 1$ and $\epsilon_1, \epsilon_2 < 0$ be constants such that*

$$\|e^{t(A_0-\overline{D_0^{A,L}}C)}\| \leq M_1 e^{\epsilon_1 t} \qquad \text{and} \qquad \|e^{t(D+CD_0^{A,L})}\| \leq M_2 e^{\epsilon_2 t}, \qquad t\geq 0.$$

If C is a bounded operator and

$$\|C\|\|B-\overline{D_0^{A,L}}(D+CD_0^{A,L})\| < \frac{\epsilon_1\epsilon_2}{M_1M_2}, \tag{2.4}$$

then the semigroup generated by $\mathbf{A}$ is uniformly exponentially stable, too.

Observe that (2.4) can be interpreted as a sufficient condition for stabilizability of the system associated with $\mathbf{A}$, if we regard B as a feedback control.

3. Two applications

Example 3.1. Let Ω be an open, bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We first show how the generation result of Section 2 can be applied in order to discuss the Laplacian on Ω equipped with so-called Wentzell-Robin (or generalized Wentzell) boundary conditions, i.e.,

$$\Delta u(z) + k\frac{\partial u}{\partial \nu}(z) + \gamma u(z) = 0, \qquad z \in \partial\Omega. \tag{WBC}$$

This problem has been tackled and already solved in three papers ([9], [2], and [19]) by quite different methods. We are going to prove the generation result by means of the abstract technique of operator matrices with coupled domain: in fact, the one-dimensional case has already been considered, also by means of operator matrices, by Kramar, Nagel, and the author in [12, § 9], thus we now focus on the case $n \geq 2$ (see also [4] for yet another approach to a similar, non-dissipative system).

It has been shown both in [9] and [2] that the correct L^2-realization of the Laplacian equipped with (WBC) is the operator matrix

$$\mathcal{A} := \begin{pmatrix} \Delta & 0 \\ -k\frac{\partial}{\partial\nu} & -\gamma I \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in H^{\frac{3}{2}}(\Omega) \times H^1(\partial\Omega) : \Delta u \in L^2(\Omega),\ \frac{\partial u}{\partial \nu} \in L^2(\partial\Omega) \text{ and } u_{|\partial\Omega} = x \right\}.$$

In order to apply the abstract results of Section 2, consider $\mathcal{A}$ as an operator matrix $\mathbf{A}$ with domain $D(\mathbf{A})$ defined as in (1.1)–(2.1). Here we let $X := L^2(\Omega)$, $\partial X := L^2(\partial\Omega)$, and $\partial Y := H^1(\partial\Omega)$. Moreover, we set

$$A := \Delta, \qquad D(A) := \{u \in H^{\frac{3}{2}}(\Omega) : \Delta u \in L^2(\Omega)\}$$

and further

$$B := 0, \qquad Cu := -a\frac{\partial u}{\partial \nu}, \qquad D(C) := D(A),$$

where $k \in \mathbb{R}$, and

$$Dx := -\gamma x, \qquad D(D) := L^2(\partial\Omega).$$

Here we have assumed that $\gamma \in L^\infty(\Omega)$. Finally, we define

$$Lu := u_{|\partial\Omega},$$

which is known to be a surjective operator from $D(A)$ to ∂Y whenever $\partial\Omega$ is smooth enough, cf. [13, Vol. I, Thm. 2.7.4]. By standard boundary regularity results we now obtain that $A_{|\ker(L)}$ is in fact the Laplacian with homogeneous Dirichlet boundary conditions, the generator of an analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(\Omega)$. Moreover, the closedness of $\binom{A}{L}$ holds by interior estimates for general elliptic operators, (a short proof of this can be found in [4, § 3]), and D is bounded whenever $\gamma \in L^\infty(\partial\Omega)$. This shows that the Assumptions 2.6 are satisfied.

In particular, by Lemma 2.3 there exists for all $\lambda \in \rho(A_0)$ the Dirichlet operator $D_\lambda^{A,L}$ associated with (A, L), a bounded operator from ∂Y to X. In fact, it is known from [13, Vol. I, $\tilde{7}$] that for all $\lambda \geq 0$ the operator $D_\lambda^{A,L}$ has a bounded extension $\overline{D_\lambda^{A,L}}$ from ∂X to X. Moreover, it also follows from [13, Vol. I, Thm. 2.7.4] that C is a bounded operator from $[D(A)_L]$ to ∂X, so that we can apply Theorem 2.9.

We still need to take a closer look to $D_0 := D + CD_0^{A,L}$: such an operator maps $H^1(\partial\Omega)$ into $L^2(\partial\Omega)$ by

$$D_0 x = -k\frac{\partial}{\partial\nu}D_0^{A,L}x - \gamma x.$$

Such an operator often occurs in the contexts of PDE's and control theory, and it is sometimes called *Dirichlet-Neumann* operator. It is known that D_0 is the operator associated to the sesquilinear form

$$\mathfrak{a}(x,y) := k\int_\Omega \nabla D_0^{A,L}x\overline{\nabla D_0^{A,L}y} + \int_{\partial\Omega}\gamma x\overline{y}, \qquad D(a) := H^{\frac{1}{2}}(\partial\Omega).$$

The form $\mathfrak{a}$ is clearly densely defined and symmetric. It is positive if (and only if) the scalar k is positive. Moreover, one can check that $\mathfrak{a}$ is also closed (if $k \neq 0$) and continuous, and in fact the associated operator D_0 is self-adjoint and dissipative: summing up, D_0 is the generator of an analytic semigroup of angle $\frac{\pi}{2}$ on $\partial X = L^2(\Omega)$ if (and only if) $k > 0$.

Further, by [13, Vol. II, (4.14.32)], we obtain that

$$\overline{D_0^{A,L}}(\partial X) \hookrightarrow H^{\frac{1}{2}}(\Omega) = [D(A_0), X]_{\frac{3}{4}},$$

and moreover

$$[D(D_0), \partial X]_{\frac{3}{4}} = H^{\frac{1}{4}}(\partial\Omega),$$

so that the operator B is actually bounded from $[D(A_0)] = H^2(\Omega) \cap H_0^1(\Omega)$ to $[D(D_0), \partial X]_{\frac{3}{4}}$. Summing up, Theorem 2.9.(3) applies and yields that the operator matrix $\mathcal{A}$ with coupled domain generates an analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(\Omega) \times L^2(\partial\Omega)$. Moreover, checking the proof of Theorem 2.9 one sees that if D_0 is not a generator (and this holds if and only if $a < 0$, since we are assuming that $n \geq 2$), then also $\mathcal{A}$ is not: we have thus partially recovered a negative result recently obtained by Vazquez and Vitillaro in the case of constant k, cf. [19, Thm. 1].

Example 3.2. As a simple application of the stability results obtained in the paper, let us now consider the initial boundary-value problem,

$$(3.1) \qquad \left\{\begin{array}{rcll} \dot{u}(t,x) &=& \Delta u(t,x) - p(x)u(t,x), & t \geq 0,\ x \in \Omega, \\ \dot{w}(t,z) &=& \Delta w(t,z) - q(z)w(t,z), & t \geq 0,\ z \in \partial\Omega, \\ w(t,z) &=& \frac{\partial u}{\partial\nu}(t,z), & t \geq 0,\ z \in \partial\Omega, \\ u(0,x) &=& f(x), & x \in \Omega, \\ w(0,z) &=& h(z), & z \in \partial\Omega, \end{array}\right.$$

which has already been discussed in [4] and [15]. Here Ω is a bounded open domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $0 \leq p \in L^\infty(\Omega)$, $0 \leq q \in L^\infty(\partial\Omega)$.

Set

$$X := L^2(\Omega), \quad \partial Y = \partial X := H^{\frac{1}{2}}(\partial\Omega).$$

Define the operators

$$Au := \Delta u - pu, \qquad u \in D(A) := H^2(\Omega),$$

$$Lu := \frac{\partial u}{\partial\nu}, \qquad u \in D(L) := D(A),$$

$$B = C = 0,$$

$$Dw := \Delta w - qw, \qquad w \in D(D) := H^{\frac{5}{2}}(\partial\Omega),$$

i.e., D is (up to a bounded perturbation) the Laplace–Beltrami operator on $\partial\Omega$. Then, $A_0 = A_{|\ker(L)}$ is (up to a bounded perturbation) the Laplacian with Neumann

boundary conditions, and one sees that the Assumptions 2.6 are satisfied, hence Theorem 2.9.(2) applies and we conclude that (3.1) is governed by an analytic semigroup on $L^2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ (in fact, as shown in [4, § 3] and [15, § 5], the problem is well-posed on the whole space $L^2(\Omega) \times L^2(\partial\Omega)$). Observe that a direct computation shows that the generator $\mathbf{A}$ of such semigroup is not dissipative.

However, if $n \geq 2$ it is known (see [13, Vol. I, Thm. 2.7.4]) that the operator $D_\lambda^{A,L}$ extends to an operator that is bounded from $H^{-\frac{3}{2}}(\partial\Omega)$ to $L^2(\Omega)$. Moreover, the Laplace–Beltrami operator D maps $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{3}{2}}(\partial\Omega)$, so that the Assumptions 2.10 are satisfied. Since both A_0 and D are dissipative and self-adjoint, the non-resonance condition of Proposition 2.14 is clearly satisfied and we conclude that the semigroup generated by $\mathbf{A}$ on $L^2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is asymptotically almost periodic as well, since A_0 and D have compact resolvent.

Now, observe that A_0 and D are invertible (hence generate uniformly exponentially stable semigroups) if (and only if) $p \neq 0 \neq q$. Summing up, we can apply Propositions 2.12.(4) and obtain that if $p \neq 0 \neq q$, then the semigroup generated by $\mathbf{A}$ is uniformly exponentially stable.

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